# Robust Controller Design for AQM and $\mathcal{H}^\infty$-Performance Analysis

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## 1 Introduction

Active Queue Management (AQM) has recently been proposed in [1] to support the end-to-end congestion control for TCP traffic regulation on the Internet. For the purpose of alleviating congestion for IP networks and providing some notion of quality of service (QoS), the AQM schemes are designed to improve the Internet applications. Earliest efforts on AQM (e.g. RED in [2]) are essentially heuristic without systematic analysis. The dynamic models of TCP ([9, 12]) make it possible to design AQM using feedback control theory. We refer to [11] for a general review of Internet congestion control.

In [12], a TCP/AQM model was derived using delay differential equations. The authors further provided a control theoretic analysis for RED where the parameters of RED can be tuned as an AQM controller [4]. In [5], a Proportional-Integral controller was developed based on the linearized model of [12]. Their controller could ensure robust stability of the closed loop system in the sense of good gain and phase margin of the PI AQM [5, 6]. A challenging nature in the design of AQM is the presence of a time delay which is called *RTT* (round trip time), and the time delays are usually time varying and uncertain. In [14], $\mathcal{H}^\infty$ optimization method was proposed for AQM controller design, which allows for parameter uncertainties of *RTT*, the number of TCP connections and available link capacity. In a similar fashion, we develop in this paper robust AQM controllers based on the $\mathcal{H}^\infty$ control techniques for SISO infinite dimensional systems [3, 16]. However, the model we considered here is a LPV system with *RTT* being the scheduling parameter. We also analyze the $\mathcal{H}^\infty$ performance for the robust controllers with respect to the uncertainty bound of the scheduling parameter *RTT*. Our results show that a smaller operating range of *RTT* results in better $\mathcal{H}^\infty$ performance of the AQM controller, which indicates that switching control among a set of robust controllers designed at selected smaller operating ranges can have better performance than a single $\mathcal{H}^\infty$ controller for the whole range. MATLAB simulations are also given to validate our design and analysis.

S. Tarbouriech et al. (Eds.): Advances in Communication Control Networks, LNCIS 308, pp. 49–63, 2004.

## 2 Mathematical Model of TCP/AQM

In [12], a nonlinear dynamic model for TCP congestion control was derived, where the network topology was assumed to be a single bottleneck with $N$ homogeneous TCP flows sharing the link. The congestion avoidance phase of TCP can be modeled as AIMD (additive-increase and multiplicative-decrease), where each positive ACK increases the TCP window size $W(t)$ by one per *RTT* and a congestion indication reduces $W(t)$ by half. Aggregating $N$ TCP flows through one congested router results in the following TCP dynamics [6, 12]:

$$\begin{aligned}\dot{W}(t) &= \frac{1}{R(t)} - \frac{W(t)}{2}\frac{W(t-R(t))}{R(t-R(t))}p(t-R(t)) \\ \dot{q}(t) &= [\frac{N(t)}{R(t)}W(t) - C(t)]^{+} \end{aligned} \tag{1}$$

where $R(t)$ is the *RTT*, $0 \leq p(t) \leq 1$ is the marking probability, $q(t)$ is the queue length at the router, and $C$ is the link capacity. Note

$$R(t) = T_p + \frac{q(t)}{C}$$

where $T_p$ is the propagation delay and $q(t)/C$ is the queuing delay.

Assume $N(t) = N$ and $C(t) = C$, the operating point of (1) is defined by $\dot{W} = 0$

$$R_0 = T_p + \frac{q_0}{C} \tag{2}$$

$$W_0 = \frac{R_0 C}{N} \tag{3}$$

$$p_0 = \frac{2}{W_0^2}. \tag{4}$$

Let $\delta q := q - q_0$ and $\delta p := p - p_0$, the linearization of (1) results in the following LPV time delay system, [6],

$$\frac{\delta q(s)}{\delta p(s)} := P_\theta(s) = \frac{K(\theta)\mathrm{e}^{-h(\theta)s}}{(T_1(\theta)s+1)(T_2(\theta)s+1)} \tag{5}$$

where

$$K(\theta) = \frac{C^3\theta^3}{4N^2} \tag{6}$$

$$T_1(\theta) = \theta \tag{7}$$

$$T_2(\theta) = \frac{C\theta^2}{2N} \tag{8}$$

$$h(\theta) = \theta \tag{9}$$

and $\theta = R(t) \in [T_p,\ T_p + q_{max}/C]$ is the scheduling parameter of (5) where $q_{max}$ is the buffer size. Note that we employ $\mathcal{L}\{f(t,\theta)|_{\theta=\theta_0}\} = f_{\theta_0}(s)$ to describe the LPV dynamic equations in Laplace domain at fixed parameter values.

## 3 $\mathcal{H}^\infty$ Controller Design for AQM

Consider the nominal system

$$P_0(s) := P_\theta(s)|_{\theta=\theta_0} = \frac{K(\theta_0)e^{-h(\theta_0)s}}{(T_1(\theta_0)s+1)(T_2(\theta_0)s+1)} \tag{10}$$

where $\theta_0 = R_0$ is the nominal *RTT*. We would like to design a robust AQM controller $C_0(s)$ for the nominal plant (10) so that

(i) $C_0(s)$ robustly stabilizes $P_\theta(s)$ for $\forall\theta \in \Theta := [\theta_0 - \Delta\theta,\ \theta_0 + \Delta\theta]$;
(ii) The closed loop nominal system has good tracking of the desired queue length $q_0$ which is a step-like signal.

Notice that the plant (5) can be written as

$$P_\theta(s) = P_0(s)(1 + \Delta P_\theta(s)) \tag{11}$$

where $\Delta P_\theta(s)$ is the multiplicative plant uncertainty.

It can be shown that an uncertainty bound $W_2^{(\theta_0,\Delta\theta)}$ satisfying

$$|\Delta P_\theta(s)|_{s=j\omega} \le |W_2^{(\theta_0,\Delta\theta)}(s)|_{s=j\omega} \quad \forall\omega \in \mathbb{R}^+ \tag{12}$$

is (see details of the derivation in Sect. 4)

$$W_2^{(\theta_0,\Delta\theta)}(s) = a + bs + cs^2 \tag{13}$$

where $a$, $b$ and $c$ are defined in (29). Note that once $\theta_0$ and $\Delta\theta$ are fixed, these coefficients are fixed.

Combining the robust stability and the nominal tracking performance condition, we come up with a two block infinite dimensional $\mathcal{H}^\infty$ optimization problem as follows:

Minimize $\gamma$, such that robust controller $C_0(s)$ is stabilizing $P_0(s)$ and

$$\left\| \begin{bmatrix} W_1(s)S_0(s) \\ W_2^{(\theta_0,\Delta\theta)}(s)T_0(s) \end{bmatrix} \right\|_\infty \le \gamma \tag{14}$$

where

$$\begin{aligned} S_0(s) &= (1 + P_0(s)C_0(s))^{-1} \\ T_0(s) &= 1 - S_0(s) = P_0(s)C_0(s)(1 + P_0(s)C_0(s))^{-1}, \end{aligned}$$

and $W_1(s) = 1/s$ is for good tracking of step-like reference inputs.

By applying the formulae given in [16] and [3], the optimal solution to (14) can be determined as follows [14]:

$$C_0(s) = \frac{\gamma(T_1(\theta_0)s+1)(T_2(\theta_0)s+1)}{cK(\theta_0)s^2} \frac{1}{1 + A(s) + F(s)} \tag{15}$$

where

$$A(s) = \frac{\beta\xi\gamma^2}{s} \tag{16}$$

and $F(s)$ is a finite impulse response (FIR) filter with time domain response

$$f(t) = \begin{cases} (\alpha+\xi-\beta\xi\gamma^2)\cos(\frac{t}{\gamma}) & \\ +(\alpha\xi\gamma+\beta\gamma-\frac{1}{\gamma})\sin(\frac{t}{\gamma}) & \text{for } t < h(\theta_0) \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

where

$$\begin{aligned} \beta &= \sqrt{x} \\ \xi &= \frac{1}{c\gamma}\sqrt{\frac{\gamma^2-a^2}{x}} \\ \alpha &= \sqrt{\frac{(b^2-2ac)\gamma^2-c^2}{c^2\gamma^2}+2\sqrt{x}-\frac{\gamma^2-a^2}{c^2\gamma^2 x}} \end{aligned} \tag{18}$$

with $x$ the unique positive root of

$$x^3 + \frac{b^2-2ac-a^2\gamma^2}{c^2\gamma^2}x^2 - (\gamma^2-a^2)\frac{(2ac-b^2)\gamma^2+c^2}{c^4\gamma^4}x - \frac{(\gamma^2-a^2)^2}{c^4\gamma^4} = 0 \tag{19}$$

The optimal $\mathcal{H}^\infty$ performance cost $\gamma$ is determined as the largest root of

$$1 - \frac{\gamma}{c}e^{-h(\theta_0)s}\frac{s}{(s+\xi)(s^2+\alpha s+\beta)}\bigg|_{s=\frac{i}{\gamma}} = 0 \tag{20}$$

Note that an internally robust digital implementation of the $\mathcal{H}^\infty$ AQM controller (15) includes a second-order term which is cascaded with a feedback block containing an FIR filter $F(s)$. The length of the FIR filter is $h(\theta_0)/T_s$, where $T_s$ is the sampling period.

## 4 Multiplicative Uncertainty Bound

In this section we derive an upper bound for the plant uncertainty. A similar analysis for a different version of the TCP/AQM linear dynamics was done in [14].

Recall (5) and (10), we have

$$
\begin{aligned}
&|P_\theta(s) - P_0(s)|_{s=j\omega} \\
&= \left| \frac{K(\theta)e^{-h(\theta)s}}{(T_1(\theta)s+1)(T_2(\theta)s+1)} - \frac{K(\theta_0)e^{-h(\theta_0)s}}{(T_1(\theta_0)s+1)(T_2(\theta_0)s+1)} \right|_{s=j\omega} \\
&= \left| \frac{K(\theta)e^{-\Delta hs}}{(T_1(\theta)s+1)(T_2(\theta)s+1)} - \frac{K(\theta_0)}{(T_1(\theta_0)s+1)(T_2(\theta_0)s+1)} \right|_{s=j\omega} \\
&\leq \left| \frac{|K(\theta)e^{-\Delta hs} - K(\theta)| + |K(\theta) - K(\theta_0)|}{|(T_1(\theta)s+1)(T_2(\theta)s+1)|} \right|_{s=j\omega} \\
&\quad + \left| \frac{K(\theta_0) - \frac{K(\theta_0)(T_1(\theta)s+1)(T_2(\theta)s+1)}{(T_1(\theta_0)s+1)(T_2(\theta_0)s+1)}}{(T_1(\theta)s+1)(T_2(\theta)s+1)} \right|_{s=j\omega} \\
&\leq K(\theta) \left| \frac{\frac{e^{\Delta hs}-1}{s}}{\frac{(T_1(\theta)s+1)(T_2(\theta)s+1)}{s}} \right|_{s=j\omega} + \left| \frac{\Delta K}{(T_1(\theta)s+1)(T_2(\theta)s+1)} \right|_{s=j\omega} \\
&\quad + K(\theta_0) \left| \frac{(T_1(\theta)T_2(\theta) - T_1(\theta_0)T_2(\theta_0))s^2 + (\Delta T_1 + \Delta T_2)s}{T(s)} \right|_{s=j\omega}
\end{aligned}
\tag{21}
$$

where

$$T(s) = (T_1(\theta)s+1)(T_2(\theta)s+1)(T_1(\theta_0)s+1)(T_2(\theta_0)s+1),$$

and

$$
\begin{aligned}
\Delta h &= h(\theta) - h(\theta_0), \qquad \Delta K = K(\theta) - K(\theta_0) \\
\Delta T_1 &= T_1(\theta) - T_1(\theta_0), \quad \Delta T_2 = T_2(\theta) - T_2(\theta_0)
\end{aligned}
$$

Note that

$$\left| \frac{e^{-\Delta hs} - 1}{s} \right|_{s=j\omega} \leq |\Delta h|$$

and

$$\left| \frac{(T_1(\theta)s+1)(T_2(\theta)s+1)}{s} \right|_{s=j\omega} \geq \max(T_1^-, T_2^-)$$

where

$$
\begin{aligned}
T_1^- &:= \min\{T_1(\theta),\ \theta \in [T_p,\ T_p + q_{max}/C]\} = T_p, \\
T_2^- &:= \min\{T_2(\theta),\ \theta \in [T_p,\ T_p + q_{max}/C]\} = \frac{CT_p^2}{2N}
\end{aligned}
$$

which are straightforward from (7) and (8). Thus

$$\left| \frac{\frac{e^{\Delta hs}-1}{s}}{\frac{(T_1(\theta)s+1)(T_2(\theta)s+1)}{s}} \right|_{s=j\omega} \leq \frac{|\Delta h|}{\max(T_1^-, T_2^-)}. \tag{22}$$

Recall

$$\begin{aligned}\Delta T_{12} &:= T_1(\theta)T_2(\theta) - T_1(\theta_0)T_2(\theta_0)\\ &= (T_1(\theta_0)+\Delta T_1)(T_2(\theta_0)+\Delta T_2) - T_1(\theta_0)T_2(\theta_0)\\ &= \Delta T_1\Delta T_2 + T_1(\theta_0)\Delta T_2 + T_2(\theta_0)\Delta T_1. \end{aligned} \tag{23}$$

We have

$$\begin{aligned}&\left|\frac{\Delta T_{12}s^2+(\Delta T_1+\Delta T_2)s}{T(s)}\right|_{s=j\omega}\\ &\leq \left|\frac{|\Delta T_{12}s^2|+|(\Delta T_1+\Delta T_2)s|}{|T(s)|}\right|_{s=j\omega}\\ &\leq \left|\frac{|\Delta T_1\Delta T_2|+|T_1(\theta_0)\Delta T_2|+|T_2(\theta_0)\Delta T_1|}{|\frac{(T_1(\theta)s+1)(T_2(\theta)s+1)(T_1(\theta_0)s+1)(T_2(\theta_0)s+1)}{s^2}|}\right|_{s=j\omega} + \frac{|\Delta T_1+\Delta T_2|}{\max(T_1(\theta_0),T_2(\theta_0))}\\ &\leq \frac{|\Delta T_1\Delta T_2|+|T_1(\theta_0)\Delta T_2|+|T_2(\theta_0)\Delta T_1|}{T_1(\theta_0)T_2(\theta_0)} + \frac{|\Delta T_1|+|\Delta T_2|}{\max(T_1(\theta_0),T_2(\theta_0))}\\ &\leq \frac{|\Delta T_1|}{T_1(\theta_0)} + \frac{|\Delta T_2|}{T_2(\theta_0)} + \frac{|\Delta T_1\Delta T_2|}{T_1(\theta_0)T_2(\theta_0)} + \frac{|\Delta T_1|+|\Delta T_2|}{\max(T_1(\theta_0),T_2(\theta_0))}\end{aligned}$$

Invoking (21) and (22), we have

$$\begin{aligned}&|P_\theta(s)-P_0(s)|_{s=j\omega}\\ &\leq K(\theta)\frac{|\Delta h|}{\max(T_1^-,T_2^-)} + |\Delta K| + K(\theta_0)(\frac{|\Delta T_1|}{T_1(\theta_0)} + \frac{|\Delta T_2|}{T_2(\theta_0)}\\ &\quad + \frac{|\Delta T_1\Delta T_2|}{T_1(\theta_0)T_2(\theta_0)} + \frac{|\Delta T_1|+|\Delta T_2|}{\max(T_1(\theta_0),T_2(\theta_0))})\end{aligned} \tag{24}$$

Defining

$$K^+ := \max\{K(\theta),\ \theta\in[T_p,\ T_p+q_{max}/C]\} = \frac{(CT_p+q_{max})^3}{4N^2},$$

and assuming

$$\begin{aligned}&|\frac{\mathrm{d}h(\theta)}{\mathrm{d}\theta}| \leq \beta_h \quad |\frac{\mathrm{d}T_1(\theta)}{\mathrm{d}\theta}| \leq \beta_{T_1}\\ &|\frac{\mathrm{d}T_2(\theta)}{\mathrm{d}\theta}| \leq \beta_{T_2} \quad |\frac{\mathrm{d}K(\theta)}{\mathrm{d}\theta}| \leq \beta_K,\end{aligned} \tag{25}$$

the additive uncertainty (24) can be rewritten as

$$\begin{aligned}&|P_\theta(s)-P_0(s)|_{s=j\omega} \leq \Delta_{(\theta_0,\Delta\theta)}\\ &:= \frac{K(\theta_0)\beta_{T_1}\beta_{T_2}}{T_1(\theta_0)T_2(\theta_0)}(\Delta\theta)^2 + (\frac{K^+\beta_h}{\max(T_1^-,T_2^-)} + \beta_K\\ &\quad + \frac{K(\theta_0)\beta_{T_1}}{T_1(\theta_0)} + \frac{K(\theta_0)\beta_{T_2}}{T_2(\theta_0)} + \frac{K(\theta_0)(\beta_{T_1}+\beta_{T_2})}{\max(T_1(\theta_0),T_2(\theta_0))})\Delta\theta.\end{aligned} \tag{26}$$

With (11) and (26), the multiplicative uncertainty $\Delta P_\theta(s)$ can be bounded by

$$|\Delta P_\theta(s)|_{s=j\omega} \leq \Delta_{(\theta_0,\Delta\theta)}|P_0(s)^{-1}|_{s=j\omega} = |W_2^{(\theta_0,\Delta\theta)}(s)|_{s=j\omega} \tag{27}$$

where

$$W_2^{(\theta_0,\Delta\theta)}(s) = a + bs + cs^2 \tag{28}$$

with

$$\begin{aligned} a &= \frac{\Delta_{(\theta_0,\Delta\theta)}}{K(\theta_0)} \\ b &= \frac{\Delta_{(\theta_0,\Delta\theta)}(T_1(\theta_0)+T_2(\theta_0))}{K(\theta_0)} \\ c &= \frac{\Delta_{(\theta_0,\Delta\theta)}T_1(\theta_0)T_2(\theta_0)}{K(\theta_0)}. \end{aligned} \tag{29}$$

## 5 $\mathcal{H}^\infty$-Performance Analysis

As shown in Sect.3, the $\mathcal{H}^\infty$ AQM controller (15) is designed for $P_\theta(s)|_{\theta=\theta_0}$ and allows for $\theta \in \Theta = [\theta - \Delta\theta,\ \theta + \Delta\theta]$. In this section, we would like to investigate the $\mathcal{H}^\infty$-performance for the corresponding closed loop system, which indicates the system robustness and system response.

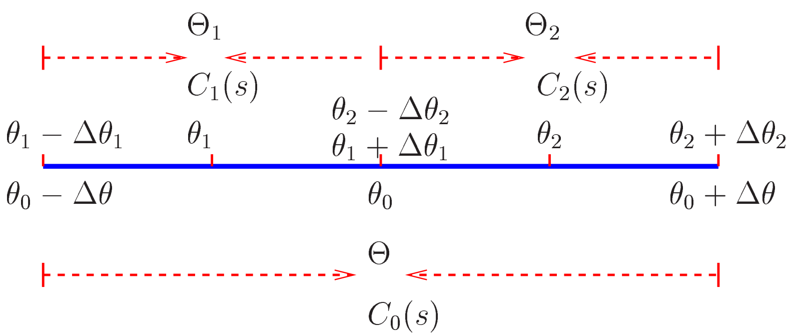


**Fig. 1.** Partition of $\Theta$ by $\Theta_1$ and $\Theta_2$

Define the $\mathcal{H}^\infty$-performance of controller $C_0(s)$ with respect to $P_\theta(s)$ as follows:

$$\gamma_{C_0}(\theta) = \left\| \begin{bmatrix} W_1(s)S(s) \\ W_2^{(\theta_0,\Delta\theta)}(s)P_0(s)C_0(s)S(s) \end{bmatrix} \right\|_\infty \tag{30}$$

for any $\theta \in \Theta = [\theta_0 - \Delta\theta,\ \theta_0 + \Delta\theta]$, where

$$S(s) = (1 + P_\theta(s)C_0(s))^{-1}, \tag{31}$$

here the term $|W_2^{(\theta_0,\Delta\theta)}(j\omega)P_0(j\omega)|$ can be seen as a bound on the additive plant uncertainty.

Furthermore, we define

$$\gamma_{C_0}^{\Delta\theta} := \sup_{\theta\in\Theta}\{\gamma_{C_0}(\theta)\} \tag{32}$$

which corresponds to the worst system response of controller $C_0(s)$ for plant $P_\theta(s)$ with $\forall\theta \in [\theta_0-\Delta\theta,\ \theta_0+\Delta\theta]$. Notice that a smaller $\gamma_{C_0}^{\Delta\theta}$ means better performance of the robust controller within the operating range $\Theta$.

Particularly, we are interested in the scenario depicted in Fig. 1, where $\Theta$ is equally partitioned by $\Theta_1 = [\theta_1-\Delta\theta_1,\ \theta_1+\Delta\theta_1]$ and $\Theta_2 = [\theta_2-\Delta\theta_2,\ \theta_2+\Delta\theta_2]$, with $\Delta\theta_1 = \Delta\theta_2 = \frac{\Delta\theta}{2}$. For $\theta \in \Theta_i$, $i = 1,2$, we design $\mathcal{H}^\infty$ controller $C_i(s)$ obeying (15) with the nominal plant $P_i(s) := P_\theta(s)|_{\theta=\theta_i}$. Similar to (30) and (32), we have

$$\gamma_{C_i}(\theta) = \left\| \begin{bmatrix} W_1(s)S_i(s) \\ W_2^{(\theta_i,\Delta\theta_i)}(s)P_i(s)C_i(s)S_i(s) \end{bmatrix} \right\|_\infty \tag{33}$$

for any $\theta \in \Theta_i \ \ i = 1,2$, and

$$\gamma_{C_i}^{\Delta\theta_i} := \sup_{\theta\in\Theta_i}\{\gamma_{C_i}(\theta)\} \quad i = 1,2 \tag{34}$$

where $S_i(s) = (1+P_\theta(s)C_i(s))^{-1}$ is defined similarly to (31).

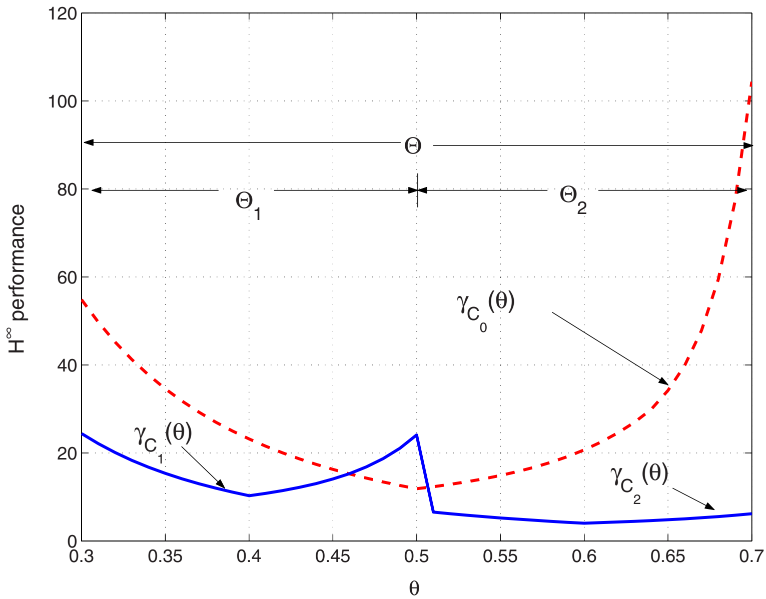


**Fig. 2.** $\mathcal{H}^\infty$ performance with respect to $\theta$

In what follows, we provide numerical analysis of the $\mathcal{H}^\infty$-performance with respect to the operating ranges and corresponding controllers shown in Fig. 1. Assume $N = 150$, $C = 500$, $\Delta\theta = 0.2$, and $\theta_0 = 0.5$, the $\mathcal{H}^\infty$ performance $\gamma_{C_0}(\theta)$ and $\gamma_{C_i}(\theta),\ i = 1,2$ can be numerically obtained from (30) and (33). As depicted in Fig. 2, it is straightforward to have

$$\max(\gamma_{C_1}^{\Delta\theta_1}, \gamma_{C_2}^{\Delta\theta_2}) = 24.4 < \gamma_{C_0}^{\Delta\theta} = 104.4$$

which means that the partition of Fig. 1 can improve system performance in the sense of smaller $\mathcal{H}^\infty$-performance cost. In fact, it is a general trend that

$$\max(\gamma_{C_1}^{\Delta\theta_1}, \gamma_{C_2}^{\Delta\theta_2}) < \gamma_{C_0}^{\Delta\theta}, \tag{35}$$

which can be further verified by Fig. 3, Fig. 4, and Fig. 5, where $N$ is chosen from 100 to 200, $C$ from 400 to 600.

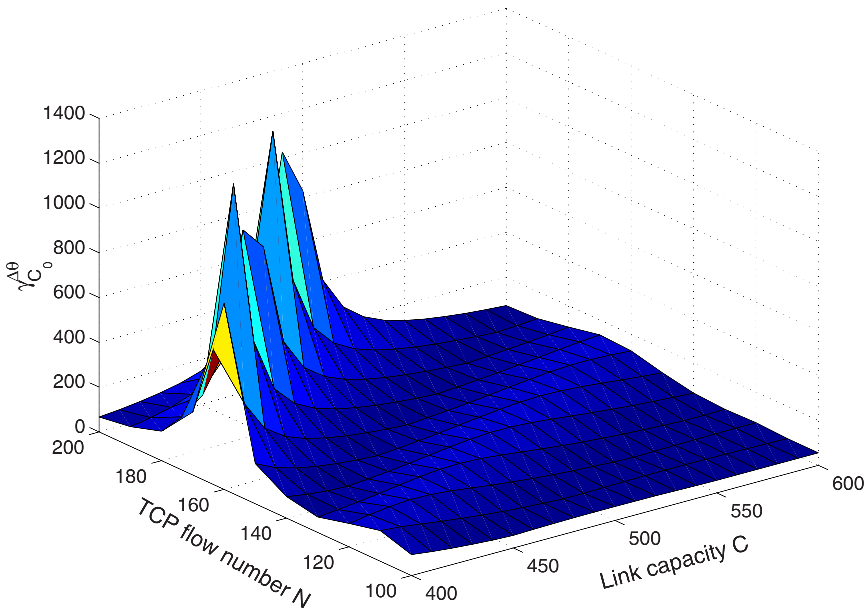


**Fig. 3.** Performance cost $\gamma_{C_0}^{\Delta\theta}$ w.r.t. $N$ and $C$

Based on the observation of better performance obtained by the partition shown in Fig. 1, it is natural to consider switching robust control among a set of $\mathcal{H}^\infty$ controllers, each of which is designed for a smaller operating range. We provide in Sect. 6 the simulation results of switching control between two robust controllers.

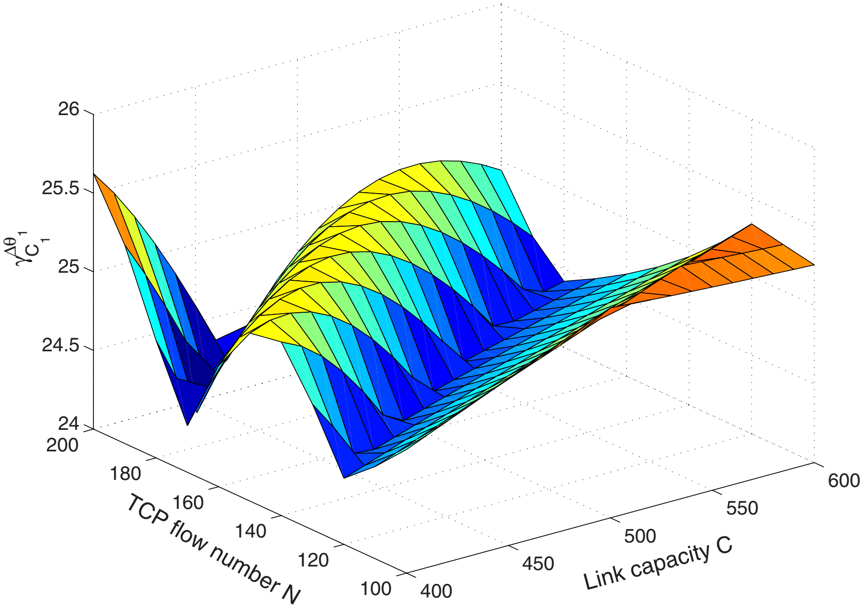


**Fig. 4.** Performance cost $\gamma_{C_1}^{\Delta\theta_1}$ w.r.t. $N$ and $C$

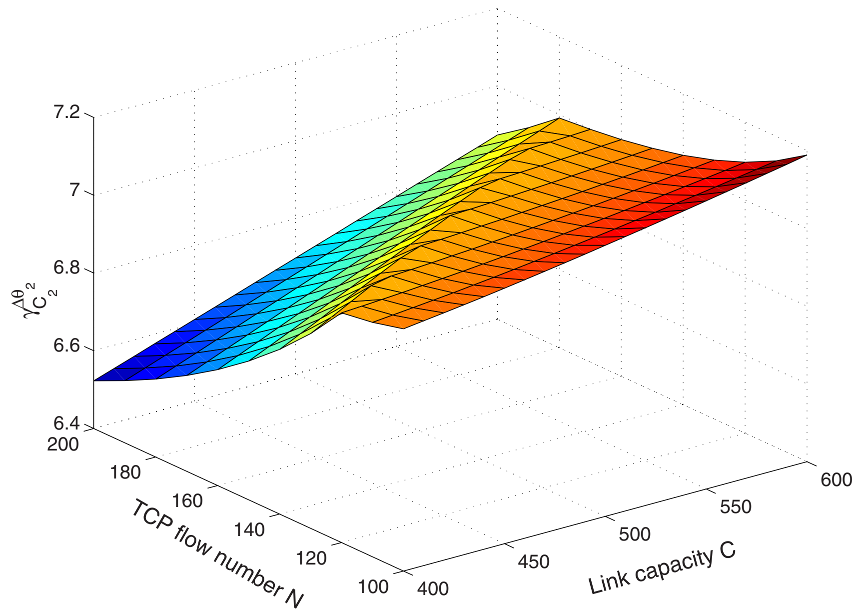


**Fig. 5.** Performance cost $\gamma_{C_2}^{\Delta\theta_2}$ w.r.t. $N$ and $C$

## 6 Simulations

The closed loop system with the determined controllers is implemented in MATLAB to validate the controller design as well as the $\mathcal{H}^\infty$ performance analyzed in previous sections. We assume the TCP flow number $N = 150$, the link capacity $C = 500$ packets/sec. The propagation delay $T_p$ is set to be 0.3 sec and the desired queue size is $q_0 = 100$ packets. Therefore, the nominal *RTT* is 0.5 sec ($\theta_0 = 0.5$), which is straightforward from (2).

### 6.1 The Case of a Single Controller

We use $\Delta\theta = 0.2$ in the design of $C_0(s)$ and $\Delta\theta_1 = \Delta\theta_2 = 0.1$ in $C_1(s)$ and $C_2(s)$. The following three scenarios are considered:

- Assuming the plant is the nominal one, i.e. $P_\theta(s) = P_0(s)$, we implement controller $C_0(s)$ as well as $C_1(s)$ and $C_2(s)$. It is shown in Fig. 6 that the three controllers can stabilize the queue length because the nominal value $\theta_0$ is within the operating range of $\Theta$, $\Theta_1$, and $\Theta_2$. Note that the system response of $C_0(s)$ is better than the other two due to the fact that it achieves the optimal $\mathcal{H}^\infty$-performance at $\theta_0$.

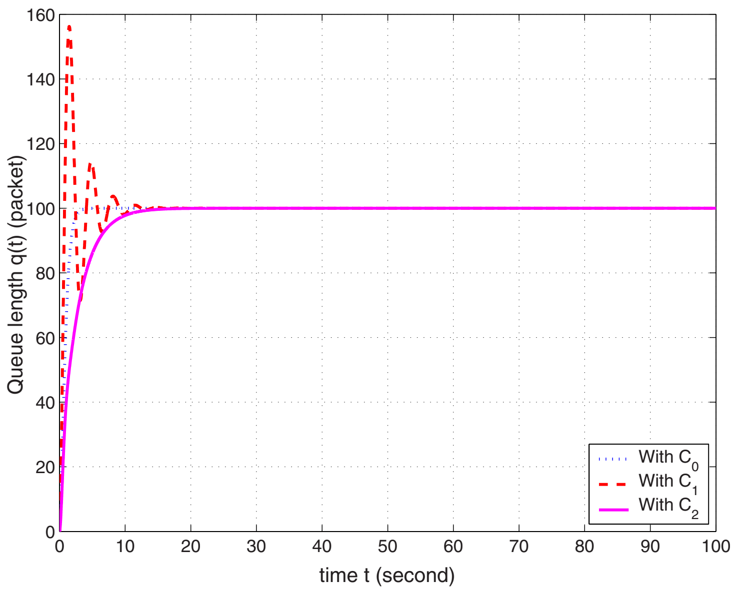


**Fig. 6.** System responses of $C_0$, $C_1$ and $C_2$ at $\theta = \theta_0 = 0.5$

- Assuming $\theta = \theta_0 - \Delta\theta = 0.3$, we implement controller $C_0$ and $C_1$ ($C_2$ is not eligible in this scenario). As depicted in Fig. 7, $C_0$ and $C_1$ can robustly stabilize the queue length. Observe that the system response of $C_1$ is better because it has much smaller $\mathcal{H}^\infty$ performance cost, which has been shown in Sect. 5.

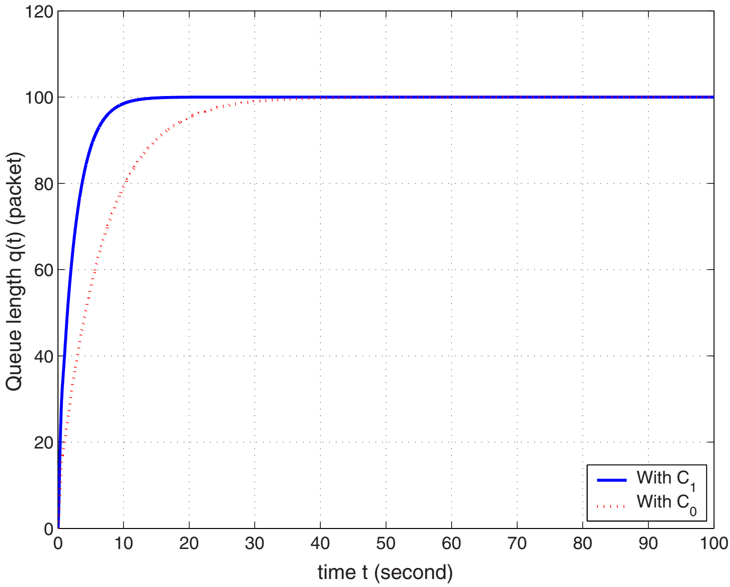


**Fig. 7.** System responses of $C_0$ and $C_1$ at $\theta = \theta_0 - \Delta\theta = 0.3$

- Similarly, we choose $\theta = \theta_0 + \Delta\theta = 0.7$ and repeat the simulation for controller $C_0$ and $C_2$ ($C_1$ is not eligible). As depicted in Fig. 8, the two controllers can robustly stabilize the queue length and their system responses coincide with the $\mathcal{H}^\infty$-performance analysis given previously.

The above simulations show that the proposed robust AQM controllers have good performance and robustness in the presence of parameter uncertainties. Meanwhile, the system responses also affirm a good coincidence with the $\mathcal{H}^\infty$ performance analysis in Sect. 5.

### 6.2 The Case of Switching Control

Motivated by the analysis in Sect. 5, we perform control switching in this experiment. We assume the same simulation configuration as Sect. 6.1 and investigate the closed loop system performance in the presence of switching between $\mathcal{H}^\infty$ controller $C_1$ and $C_2$ for a slow time varying signal $\theta(t) \in \Theta$. For the purpose of comparison, we also provide the system response with a single $\mathcal{H}^\infty$ controller $C_0$. As depicted in Fig. 9 and Fig. 10, the switching control method has better transient behavior in terms of smaller overshoot, faster convergence and less oscillations. Note that the large oscillations around 90 *sec* on both plots are due to the fact that $\theta$ is not assumed to

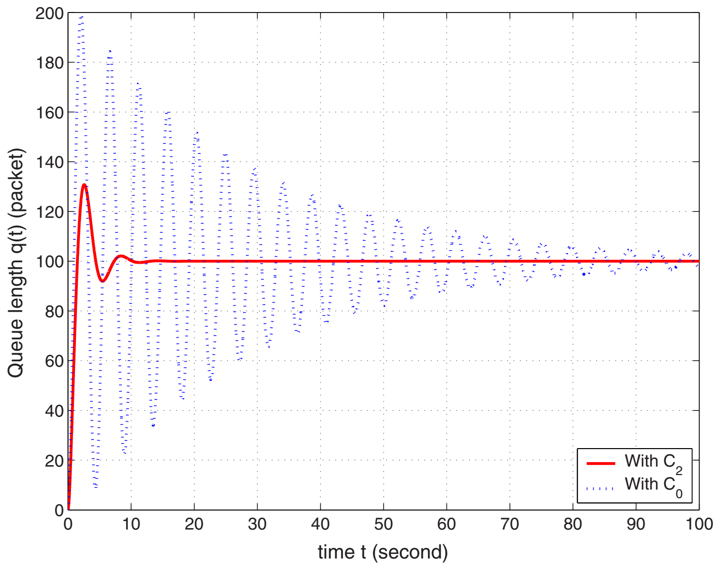


**Fig. 8.** System responses of $C_0$ and $C_2$ at $\theta = \theta_0 + \Delta\theta = 0.7$

be time varying in the proposed design. Instead, we assume it is piece-wise constant but uncertain in the derivation of the system uncertainty bound (see Section 3 and 4 for details).

## 7 Conclusions

We provided in this paper the guidelines of designing robust controllers for AQM, where the $\mathcal{H}^\infty$ techniques for infinite dimensional systems were implemented. The $\mathcal{H}^\infty$-performance was numerically analyzed with respect to the bound of the scheduling parameter $\theta$. It was shown that smaller uncertainty bound could result in better $\mathcal{H}^\infty$-performance of the corresponding closed loop systems. Accordingly, we proposed switching control between two robust controllers which outperforms a single controller. Simulations were conducted to validate the design and analysis.

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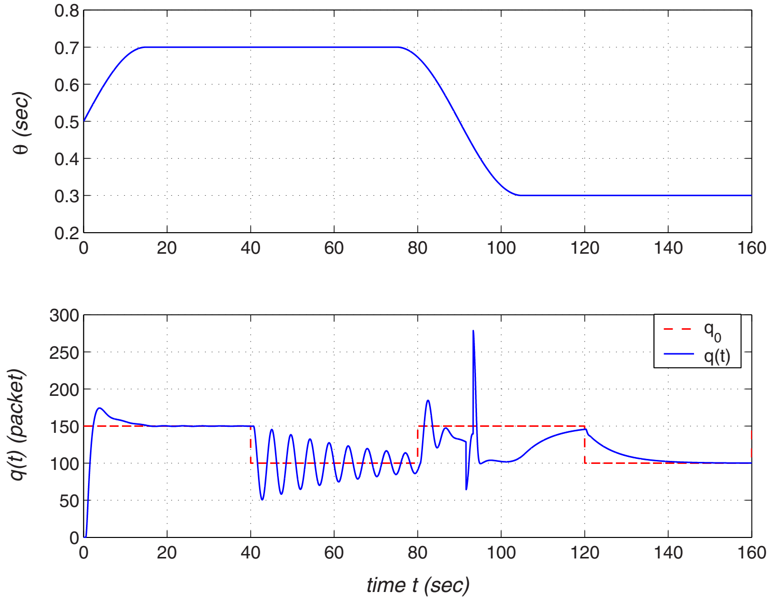


**Fig. 9.** A single controller $C_0$

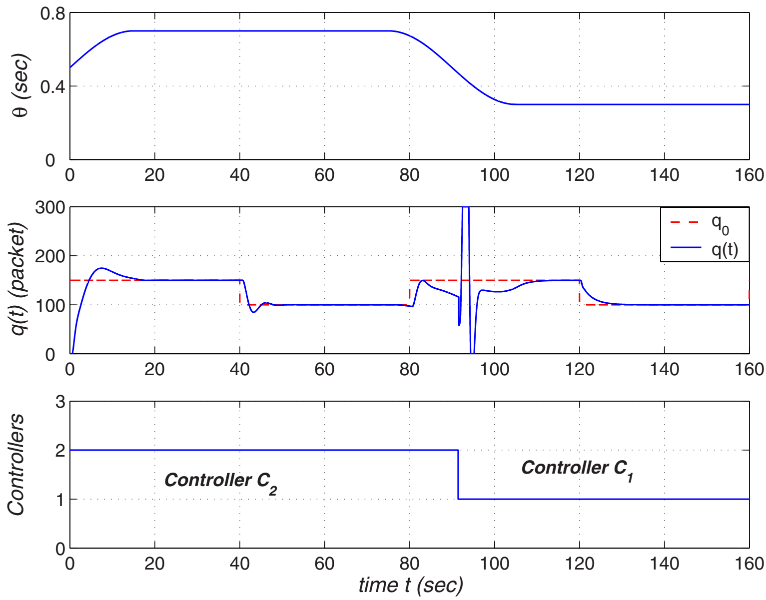


**Fig. 10.** Switching control between $C_1$ and $C_2$

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